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Exact, self-similar, time-dependent free surface flows under gravity

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A class of exact, self-similar, time-dependent solutions describing free surface flows under gravity is found which extends the self-propagating class of solutions discovered earlier by Freeman (1972) to those which decay with time.

1. Introduction

The equations describing time-dependent inviscid fluid motions with a free surface present severe difficulties owing to nonlinear boundary conditions and the exact solutions for such flows have been quite rare. The recent work in this area is due to Longuet-Higgins (1972, 1976), who has also summarized previous work on the subject. Even though the flows he has considered are governed by the full inviscid equations, he has not taken the important effects of gravity into account.

In a different class of flows under the so-called long-wave approximation, one of the first studies is due to Burns (1953) who clearly set out the shallow water equations governing sheared flows with a free surface, for which gravity plays an essential role. He wrote out the nonlinear characteristic equations that describe waves on a linearly sheared undisturbed flow and discussed in some detail the linearized version of these equations. Burns' work was followed later by Blythe, Kazakia & Varley (1972) who introduced generalized Riemann invariants for shear flows. They gave explicit large amplitude progressive wave solutions for the linear shear flow, briefly discussed earlier by Burns (1953). In a paper immediately following Blythe *et al.*, Freeman (1972) gave a class of exact similarity solutions describing simple waves on shear flows. These are self-propagating nonlinear waves whose speed in the horizontal direction is proportional to $(\tilde{g}H)^{\frac{1}{2}}$, where H is the depth of water in the channel and $\tilde{g}$, the acceleration due to gravity. By an ingenious transformation, the final solution was expressed in terms of incomplete β functions.

In the present paper, we give a class of exact, self-similar, (explicitly) time-dependent solutions of inviscid, shallow water equations, which generalize those solutions of Freeman (1972) which are self-similar. This class is similar to Longuet-Higgins (1972) with the difference noted above, namely, these solutions, in the long-wave approximation, describe gravity waves while the gravity effects are absent in the solutions discovered by Longuet-Higgins for the two-dimensional and three-dimensional fluid equations without the simplification introduced by the long-wave approximation. In our study we seek directly self-similar solutions of the governing equations, an approach somewhat different from Freeman's, though his solutions provided a useful guide and an important check on the newer class that we have found.

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Section 2 gives the equations of motion. In § 3 the self-similar equations are derived and their solutions are found. The results and conclusions are contained in § 4.

2. Equations of motion

The unsteady, two-dimensional inviscid flow equations in the 'hydraulic approximation' are

$$u_t + uu_x + vu_y + \tilde{g}H_x = 0, \tag{2.1}$$

$$u_x + v_y = 0, \tag{2.2}$$

where u, v are velocity components in the x and y directions respectively. The former coincides with the uniform horizontal bottom and the latter with the vertical direction. The so-called hydraulic approximation in the above equations is manifested in the replacement in (2.1) of the pressure p by the uniform gravitational pressure.

This approximation implies the shallow water approximation $H_0/L \ll 1$, where H_0 is a characteristic depth and L a typical wavelength.

The surface boundary condition now becomes

$$v = H_t + uH_x \quad \text{on} \quad y = H. \tag{2.3}$$

The second condition that the pressure is constant on the free surface has been incorporated in the hydraulic approximation $p = p_0 + \tilde{g}(H-y)$.

At the bottom,

$$v = 0 \quad \text{on} \quad y = 0. \tag{2.4}$$

Equations (2.1) and (2.2) govern the two unknown functions $u(x, y, t)$ and $v(x, y, t)$ while $H(x, t)$ should be found such that the boundary conditions (2.3) and (2.4) are satisfied.

To simplify the boundary conditions we change the independent and dependent variables:

$$z = \frac{y}{H(x,t)}, \quad w = \frac{Dz}{Dt} = \frac{1}{H}\{v - z(H_t + uH_x)\}. \tag{2.5}$$

In terms of these variables equations (2.1) and (2.2) become

$$H_t + uH_x + H(u_x + w_z) = 0, \tag{2.6}$$

$$u_t + uu_x + wu_z + gH_x = 0, \tag{2.7}$$

while the boundary conditions (2.3) and (2.4) simplify to

$$w = 0 \quad \text{on} \quad z = 0, 1. \tag{2.8}$$

3. Self-similar solutions

We look for solutions of (2.6)–(2.8) in the form

$$H = t^{a_1} h(\xi), \tag{3.1}$$

$$u = t^{b_1} f(\xi, \eta), \tag{3.2}$$

$$w = t^{c_1} g(\xi, \eta), \tag{3.3}$$

where $$\xi = xt^{\alpha_1}, \quad \eta = z.$$

The choice of $\eta = z$ was dictated by the form of boundary conditions (2.8). A simple calculation shows that the self-similar form (3.1)–(3.3) is possible only if

$$a_1 = -2\alpha, \quad b_1 = -\alpha, \quad c_1 = -1, \quad \alpha_1 = \alpha - 1, \tag{3.4}$$

where α is a parameter. We restrict our attention to solutions which are either self-propagating or decay with time so that $\alpha \geqslant 0$. Equations (2.6) and (2.7) now assume the form

$$f_\xi + g_\eta + [(\alpha-1)\xi + f]\frac{1}{h}\frac{dh}{d\xi} - 2\alpha = 0, \tag{3.5}$$

$$gf_\eta + [(\alpha-1)\xi + f]f_\xi - \alpha f + \tilde{g}\frac{dh}{d\xi} = 0. \tag{3.6}$$

Multiplying (3.5) by $(\alpha-1)\xi + f$ and (3.6) by h, and subtracting, we get, after some rearrangement,

$$\frac{((\alpha-1)\xi + f)g_\eta - gf_\eta}{[(\alpha-1)\xi+f]^2} + \frac{1}{h}\frac{dh}{d\xi} + \frac{\alpha f - \tilde{g}\,dh/d\xi}{[(\alpha-1)\xi+f]^2} - \frac{2\alpha}{(\alpha-1)\xi+f} = 0. \tag{3.7}$$

Integrating (3.7) with respect to η and using the boundary condition $g = 0$ at $\eta = 0$ corresponding to $w = 0$ at $z = 0$, we get

$$\frac{g}{(\alpha-1)\xi+f} + \frac{1}{h}\frac{dh}{d\xi}\eta + \int_0^\eta \left(\frac{-2\alpha}{(\alpha-1)\xi+f} + \frac{\alpha f - \tilde{g}h'}{((\alpha-1)\xi+f)^2}\right)d\eta = 0, \tag{3.8}$$

where $h' = dh/d\xi$. The second boundary condition $g = 0$ at $\eta = 1$, corresponding to $w = 0$ at $\eta = 1$, leads to

$$\left(\frac{h'}{h}\right)^{-1}\int_0^1 \left(\frac{2\alpha}{(\alpha-1)\xi+f} + \frac{\tilde{g}h' - \alpha f}{((\alpha-1)\xi+f)^2}\right)d\eta = 1. \tag{3.9}$$

This is a generalization of the Burns' condition to nonlinear time-dependent flows. After a cumbersome manipulation it is possible to derive the following partial differential equation for

$$\begin{aligned} N &= (-1)[2\bar{\alpha}(\alpha-1) + U]^{-1} \\ &\equiv (-1][2\bar{\alpha}(\alpha-1) + 2\bar{\alpha})\xi^{-1}f]^{-1}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} N_\eta N_{\xi\eta} &+ \left[\frac{2\alpha-1}{(\tilde{g}h)^{\frac{1}{2}}}N^2 + \left(\frac{h'}{h} + \frac{\alpha(\alpha-1)\xi}{\tilde{g}h}\right)N^3 - N_\xi + \frac{N}{2}\frac{h'}{h}\right]N_{\eta\eta} \\ &- \left[\frac{5\alpha-2}{(\tilde{g}h)^{\frac{1}{2}}}N + \frac{3}{2}\frac{h'}{h} + 2\left(\frac{h'}{h} + \frac{\alpha(\alpha-1)\xi}{\tilde{g}h}\right)N^2\right]N_\eta^2 = 0. \end{aligned} \tag{3.11}$$

Here a parameter $\bar{\alpha}$ is defined by

$$\xi = 2\bar{\alpha}(\tilde{g}h)^{\frac{1}{2}}. \tag{3.12}$$

Since we shall be mainly interested in flows for which N is a function of η only, the direct substitution

$$f = (2\bar{\alpha})^{-1}\xi U, \quad g = (2\bar{\alpha})^{-1}G, \quad \xi = 2\bar{\alpha}(\tilde{g}h)^{\frac{1}{2}}, \tag{3.13}$$

in (3.5) and (3.6) and elimination of G immediately leads to†

$$[(1+2\alpha\bar{\alpha}^2(\alpha-1))N^3+\bar{\alpha}(2\alpha-1)N^2+\tfrac{1}{2}N]\frac{d^2N}{d\eta^2} - [2(1+2\alpha(\alpha-1)\bar{\alpha}^2)N^2+(5\alpha-2)\bar{\alpha}N+\tfrac{3}{2}]\left(\frac{dN}{d\eta}\right)^2 = 0. \quad (3.14)$$

Equation (3.14) can be integrated to give

$$\eta = -C\int \frac{(N_1-N)^{-(2\alpha\bar{\alpha}N_1+1)/2[\bar{\alpha}(1-2\alpha)N_1-1]}(N-N_2)^{-(2\alpha\bar{\alpha}N_2-1)/(2[\bar{\alpha}(1-2\alpha)N_2-1]}}{N^3}\,dN, \quad (3.15)$$

where C is a constant and N_1 and N_2 are the roots of the quadratic equation

$$[1+2\alpha\bar{\alpha}^2(\alpha-1)]N^2+\bar{\alpha}(2\alpha-1)N+\tfrac{1}{2}=0$$

given by

$$N_{1,2} = \frac{(1-2\alpha)\bar{\alpha}\pm(\bar{\alpha}^2-2)^{\frac{1}{2}}}{2[1+2\alpha(\alpha-1)\bar{\alpha}^2]}. \quad (3.16)$$

We assume that N_1, N_2 are real and, therefore, $\bar{\alpha}^2 > 2$. Besides, we restrict ourselves to flows which do not contain a critical level where particle speed is equal to wave speed, so that $f \neq (1-\alpha)\xi$ anywhere in the range of f. The (normalized) velocity

$$U = f/(\tilde{g}h)^{\frac{1}{2}} = 2\bar{\alpha}(1-\alpha)-1/N$$

varies from $U_1 = 2\bar{\alpha}(1-\alpha)-1/N_2$ at the bottom $\eta = 0$ to $U_2 = 2\bar{\alpha}(1-\alpha)-1/N_1$ at the surface $\eta = 1$, where $N_1 > N_2$. With these boundary conditions, equation (3.15) can be written as

$$\eta = \frac{\displaystyle\int_0^{U+1/N_2-2\bar{\alpha}(1-\alpha)}\left(\frac{1}{N_2}-\frac{1}{N_1}-z\right)^{-(1+2\alpha\bar{\alpha}N_1)/[2\bar{\alpha}(1-2\alpha)N_1-1]} z^{(1+2\alpha\bar{\alpha}N_2)/[2\bar{\alpha}(1-2\alpha)N_2-1]}\,dz}{\displaystyle\int_0^{1/N_2-1/N_1}\left(\frac{1}{N_2}-\frac{1}{N_1}-z\right)^{-(1+2\alpha\bar{\alpha}N_1)/[2\bar{\alpha}(1-2\alpha)N_1-1]} z^{(1+2\alpha\bar{\alpha}N_2)/[2\bar{\alpha}(1-2\alpha)N_2-1]}\,dz}, \quad (3.17)$$

which can be expressed in terms of beta functions as

$$\eta = \frac{B\left(\dfrac{2\bar{\alpha}(1-4\alpha)N_2-3}{2\bar{\alpha}(1-2\alpha)N_2-1}, \dfrac{2\bar{\alpha}(1-4\alpha)N_1-3}{2\bar{\alpha}(1-2\alpha)N_1-1}, \dfrac{U+1/N_2-2\bar{\alpha}(1-\alpha)}{1/N_2-1/N_1}\right)}{B\left(\dfrac{2\bar{\alpha}(1-4\alpha)N_2-3]}{2[\bar{\alpha}(1-2\alpha)N_2-1]}, \dfrac{2\bar{\alpha}(1-4\alpha)N_1-3}{2[\bar{\alpha}(1-2\alpha)N_1-1]}, 1\right)} \quad (3.18)$$

$$\left.\begin{aligned} \text{provided } \mu &\equiv 1-\frac{1}{2}\frac{1+2\alpha\bar{\alpha}N_1}{\bar{\alpha}(1-2\alpha)N_1-1} > 0,\\ \nu &\equiv 1-\frac{1}{2}\frac{1+2\alpha\bar{\alpha}N_2}{\bar{\alpha}(1-2\alpha)N_2-1} > 0.\end{aligned}\right\} \quad (3.19)$$

We substitute the velocity profile (3.17) in

$$\begin{aligned} I &= \left(\frac{h'}{h}\right)^{-1}\int_0^\eta\left(\frac{2\alpha}{(\alpha-1)\xi+f}+\frac{\tilde{g}h'-\alpha f}{((\alpha-1)\xi+f)^2}\right)d\eta \\ &= \frac{1}{D}\int_{U_1}^{U}\left(\frac{\alpha\bar{\alpha}}{2\bar{\alpha}(\alpha-1)+U}+\frac{1+2\alpha(\alpha-1)\bar{\alpha}^2}{[2\bar{\alpha}(\alpha-1)+U]^2}\right) \\ &\quad\times\left(2\bar{\alpha}(1-\alpha)-\frac{1}{N_1}-U\right)^{-(1+2\alpha\bar{\alpha}N_1)/[2\bar{\alpha}(1-2\alpha)N_1-1]} \\ &\quad\times\left(U+\frac{1}{N_2}-2\bar{\alpha}(1-\alpha)\right)^{-(1+2\alpha\bar{\alpha}N_2)/[2\bar{\alpha}(1-2\alpha)N_2-1]}dU, \end{aligned} \quad (3.20)$$

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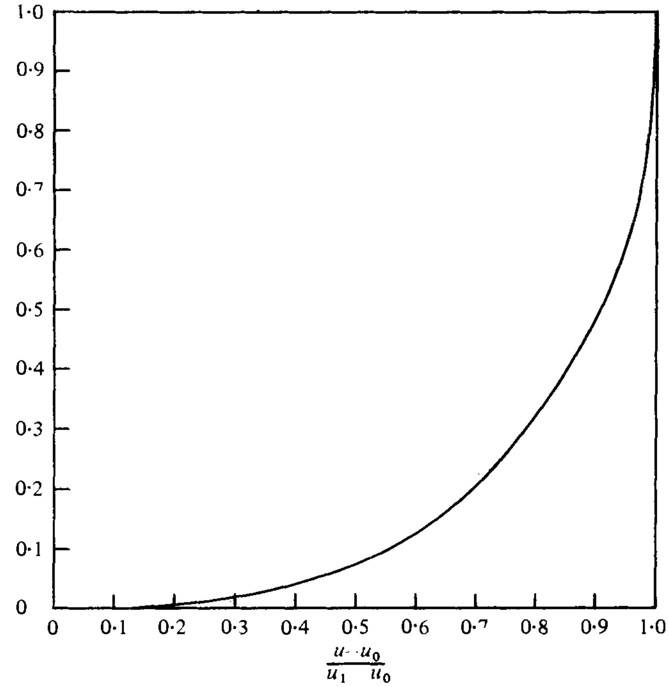

FIGURE 1. A typical velocity profile with $\alpha = 0{\cdot}06$ and $\bar{\alpha} = 1{\cdot}7$.

where

$$D = \int_0^{1/N_2 - 1/N_1} \left(\frac{1}{N_2} - \frac{1}{N_1} - z\right)^{-\frac{1}{2}(1+2\alpha\bar{\alpha}N_1)/[\bar{\alpha}(1-2\alpha)N_1 - 1]} z^{-\frac{1}{2}(1+2\alpha\bar{\alpha}N_2)/[\bar{\alpha}(1-2\alpha)N_2 - 1]}\, dz \qquad (3.21)$$

and U_1 is the value of U at $\eta = 0$. With some effort, it may now be verified that $I(1) = 0$ so that Burns' condition (3.9) is satisfied.

4. Results and conclusions

Now we discuss the special time-dependent solutions found above which do not contain critical levels and which are in the neighbourhood of some of Freeman's solutions. The roots N_1 and N_2 of the quadratic

$$(1 + 2\alpha\bar{\alpha}^2(\alpha - 1))\, N^2 + \bar{\alpha}(2\alpha - 1)\, N + \tfrac{1}{2} = 0 \qquad (4.1)$$

are real, distinct and positive if

$$2 < \bar{\alpha}^2 < 1/2\alpha(1-\alpha), \quad 0 \leqslant \alpha < \tfrac{1}{2}. \qquad (4.2)$$

If we observe the relation

$$\frac{1}{N_2} - \frac{1}{N_1} = 2(\bar{\alpha}^2 - 2)^{\frac{1}{2}}, \qquad (4.3)$$

we can easily prove that the conditions (3.19) are satisfied if $\bar{\alpha} > 1{\cdot}5$. The value of $\bar{\alpha} = 1{\cdot}5$ corresponds to nonsheared flows. Subject to conditions (4.2) and $\bar{\alpha} > 1{\cdot}5$, we have a large variety of time-dependent self-similar solutions for which the wave speeds $\bar{\alpha}$ are limited to the range $2{\cdot}25 < \bar{\alpha}^2 < 1/2\alpha(1-\alpha)$, $0 \leqslant \alpha < \frac{1}{2}$. The upper range for

$\bar{\alpha}^2$ tends to infinity as α tends to zero, the case considered by Freeman. On the other hand as $\alpha \to \frac{1}{3}$, such waves cease to exist. Freeman's solutions are self-propagating with a speed proportional to $(\tilde{g}h)^{\frac{1}{2}}$ and do not decay with time. Besides, the dependence of h on x and t remains arbitrary for his solutions. In the present paper, while the phase function ξ is still proportional to $(gh)^{\frac{1}{2}}$, the height $H(x, t)$ is now explicitly given by $H = x^2/4\bar{\alpha}^2t^2$ and the solutions decay with time; for example, the water height decays like t^{-2}, the velocities u and w decay like $t^{-\alpha}$ and t^{-1} with $0 \leqslant \alpha < \frac{1}{3}$. This class resembles that found by Longuet-Higgins (1972, 1976). That the self-propagating or stationary solutions can in general be put in the familiar similarity form has been clearly brought out by Barenblatt & Zel'dovich (1971). This, in the present case, has already been demonstrated by comparison with Freeman's solutions.

The explicit nonlinear solution found by Blythe *et al.* (1972) for linear shear flow is a centred expansion wave, depending on x/t only, and is similar to Freeman's. The comparison with Longuet-Higgins' (1976) parabolic free surface flow is not quite direct since there is no horizontal bottom. We merely note that the dimensions of the two-dimensional free parabolic surface for his gravity-free flows vary like t^{-3} in contrast to our free parabolic surface, $H = x^2/4\bar{\alpha}^2t^2$, which under gravity and above a horizontal bottom, varies like t^{-2}. The present solutions, like those of Longuet-Higgins, are local since H would become infinitely large far away.

A typical velocity profile for $\alpha = 0{\cdot}06$ and $\bar{\alpha} = 1{\cdot}7$, corresponding to the parameters $p = 2{\cdot}5$ and $q = 0{\cdot}5$ of beta function $I_x(p, q)$, is shown in figure 1.

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